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# Global Solutions for the KdV Equation with Unbounded Data

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## 1. INTRODUCTION

This paper is concerned with the initial value problem (IVP) associated to the Korteweg–de Vries (KdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.1}$$

The IVP (1.1) has been extensively studied. In particular, its local and global solvability in the classical Sobolev spaces $H^s(\mathbb{R})$, and its weighted versions have recently received a lot of attention, (for a partial list references see [4]). In the global setting, J. Bourgain [2] showed that (1.1) is globally well posed in $u_0 \in L^2(\mathbb{R})$. In [4] we simplified this $L^2$ result, and extended it, in the local setting, to Sobolev spaces with negative index, i.e. $s < 0$. Roughly, the arguments in [2, 4] combine the dispersive part of the equation in (1.1) modeled by the linear part together with the structure of the nonlinear term.

Our interest here is on the global solvability of the IVP (1.1) with smooth and unbounded data $u_0$. This problem was suggested by a question raised to us by E. Witten [7]. In this regard we find the following result due to A. Menikoff [6]:



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THEOREM 1.1. *If $u_0(x)$ satisfies*

$$\frac{d^j}{dx^j} u_0(x) = o(|x|^{1-j}) \qquad as \quad |x| \to \infty, \qquad 0 \leqslant j \leqslant 7, \tag{1.2}$$

*then the IVP* (1.1) *has a unique global classical solution.*

(For further results in the sub-linear and linear cases we refer to [1] and references therein).

The following example shows that the growth condition in (1.2) is optimal.

For data

$$u_0(x) = -x, \tag{1.3}$$

the IVP (1.1) has a local solution

$$u(x, t) = -\frac{x}{1-t}, \tag{1.4}$$

which blows up, at any point $x \in \mathbb{R}$, at the time $t = 1$.

The same example (1.3)–(1.4) solves the IVP associated to the inviscid Burgers' equation

$$\begin{cases} \partial_t \phi + \phi \partial_x \phi = 0, & t > 0, \quad x \in \mathbb{R}, \\ \phi(x, 0) = \phi_0(x), \end{cases} \tag{1.5}$$

thus one can conclude that the dispersive effects in this case are neglected.

In the same vain one has for data

$$u_0(x) = x, \tag{1.6}$$

the IVP (1.1) (and (1.5)) has a global solution ($t > 0$)

$$u(x, t) = \frac{x}{1+t}. \tag{1.7}$$

Hence, from Theorem 1.1 and the examples (1.3)–(1.7) one may infer that the global solvability of the IVP (1.1) with data growing linearly or faster at infinity depends on its monotonicity, i.e. on the sign of its derivative.

Our main result in this paper shows that for data $u_0(x)$ of the form

$$u_0(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 + f(x), \tag{1.8}$$

with $k$ odd, $a_k > 0$, and $f \in \mathscr{S}(\mathbb{R})$ (Schwartz class) the IVP (1.1) has a global smooth solution.

THEOREM 1.2. *For any data $u_0(x)$ in the class described in* (1.8) *the IVP* (1.1) *has a unique global solution $u(x, t)$ satisfying*

$$u \in C^\infty(\mathbb{R} \times [0, \infty)), \tag{1.9}$$

*for any $T > 0$*

$$|u(x, t)| \leqslant c_T + \frac{|x|}{t}, \qquad (x, t) \in \mathbb{R} \times (0, T), \tag{1.10}$$

$$|\partial_x u(x, t)| \leqslant c_T + \frac{1}{t}, \qquad (x, t) \in \mathbb{R} \times (0, T), \tag{1.11}$$

*and*

$$\int_0^T \left( \int_{-\infty}^{\infty} |(1 + |x|^{j-2})\, \partial_x^j u(x, t)|^2 \, dx \right)^{1/2} dt \leqslant c(T; j), \qquad j = 2, 3.... \tag{1.12}$$

From the proof of Theorem 1.2 it will be clear that the same proof applies to a larger class of data. Although we do not have any result when $k$ in (1.8) is even, we believe that in that case the IVP (1.1) does not have a smooth global solution.

The behavior of the solution of (1.1) described in (1.9)–(1.12) reflects a balance between the dispersive (linear part) and the nonlinear part of the equation. On one hand, it is easy to see that solutions of the IVP (1.5) with data satisfying (1.8) may develop singularities, in its first derivative, in finite time, which violates (1.9). In fact, it does except in the cases where $u_0'(x) \geqslant 0$ for all $x \in \mathbb{R}$. On the other hand, (1.10)–(1.12) show the solution behaves as a global solution of the Burgers' equation and not as a solution of the associated linear problem. More precisely, consider the IVP for the associated linear (dispersive) equation

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & t > 0, \quad x \in \mathbb{R}, \\ v(x, 0) = x^5, \end{cases} \tag{1.13}$$

with solution $v(\cdot)$ given by the formula

$$v(x, t) = x^5 - 60x^3 t + 180t^2 \tag{1.14}$$

which does not satisfy any of the inequalities in (1.10)–(1.12). However, as it will be shown in Theorem 2.1 the solution for the associated nonlinear problem (1.5) corresponding to data $\phi_0(x) = x^5$ satisfies (1.10)–(1.12) with $c_T = 0$.

In the proof of Theorem 1.2 we first split the initial data $u_0$ as

$$u_0(x) = h(x) + w_0(x). \tag{1.15}$$

where $h(\cdot)$ is a smooth, nondecreasing function with $h(x) \sim x^k$, (see (2.2)), and $w_0 \in \mathscr{S}(\mathbb{R})$.

The IVP (1.5) with $\phi_0(x) = h(x)$ has a unique global smooth solution $\phi(x, t)$. Thus the problem has been reduced to show that the IVP

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(\phi w) + \partial_x^3 \phi = 0, & t > 0, \quad x \in \mathbb{R}, \\ w(x, 0) = w_0(x), \end{cases} \tag{1.16}$$

has a unique global smooth solution $w(x, t)$, since $u(x, t) = \phi(x, t) + w(x, t)$.

The equation in (1.16) can be seen as a KdV equation perturbed with a linear term $\partial_x(\phi w)$ and an external force $\partial_x^3 \phi$. However, these terms are "unbounded." To overcome this problem we need two key ingredients. First, the global solution $\phi(\cdot)$ of the IVP (1.5) associated to the inviscid Burgers equation decays very fast, (see Theorem 2.1). In particular, we show that for any $T > 0$

$$\|\partial_x^j \phi\|_{L^1_T L^2_x} = \int_0^T \left( \int_{-\infty}^{\infty} |\partial_x^j \phi(x, t)|^2 \, dx \right)^{1/2} dt < c_T, \qquad j = 2, 3 \ldots \tag{1.17}$$

Second, in all the needed estimates the terms involving $\phi(\cdot), \partial_x \phi(\cdot)$ appear with the appropriate sign. These facts allow us to use the classical energy estimates to solve the IVP (1.16) locally. To extend this local solution to any time interval we need further decay properties (see (2.6)) of the solution of (1.5), as well as weighted estimates of the local solution of (1.16). Once both have been established we can handle the integration by parts involved in the use of the conservation laws for the KdV equation (the second, third, and fourth, see [6]). These provide an *a priori* estimate for the local solution of (1.16), which allows us to extend it to any time interval.

This paper is organized as follows. In Section 2 we deduce all the estimates for the solution, and its derivatives, of the IVP (1.5) needed in the paper. Section 3 is concerned with the local solvability of the IVP (1.16). It also contains the weighted estimates used in Section 4 to provide the *a priori* estimate which implies the desired global solvability. In Section 4 we prove Theorem 1.2. Finally, in Section 5 we state a version of Theorem 1.2 for larger class of data than in (1.8), whose proof follows the argument in the previous section.

## 2. ESTIMATES FOR THE IVP (1.5)

In this section we shall deduce several estimates describing the behavior of the solution, and its derivatives, of the IVP for the inviscid Burgers equation

$$\begin{cases} \partial_t \phi + \phi \partial_x \phi = 0, & t > 0, \quad x \in \mathbb{R}, \\ \phi(x, 0) = h(x), \end{cases} \tag{2.1}$$

where the initial data $h(\cdot)$ is a smooth, nondecreasing function with $h(0) = 0$, and $h(x) \sim x^k$ outside the interval $[-M, M]$, i.e. for $|x| \geqslant M$

$$\begin{cases} c_j^{-1} \, |x|^{k-j} \leqslant |\partial_x^j h(x)| \leqslant c_j \, |x|^{k-j}, & j = 0, 1, ..., k-1, \\ |\partial_x^j h(x)| \leqslant c_j, & j = k, .... \end{cases} \tag{2.2}$$

THEOREM 2.1. *Under the above assumptions on the initial data $h(\cdot)$ the IVP* (2.1) *has a unique global solution $\phi(x, t)$ satisfying*

$$\phi \in C^\infty(\mathbb{R} \times [0, \infty)), \tag{2.3}$$

$$|\phi(x, t)| \leqslant \frac{|x|}{t}, \qquad \textit{for any} \quad (x, t) \in \mathbb{R} \times (0, \infty), \tag{2.4}$$

$$0 \leqslant \partial_x \phi(x, t) \leqslant \frac{1}{t}, \qquad \textit{for any} \quad (x, t) \in \mathbb{R} \times (0, \infty), \tag{2.5}$$

*and for any $j = 2, 3...$*

$$\int_0^T \|(1 + |x|^{j-2}) \, \partial_x^j \phi(\cdot, t)\|_{L^2} \, dt \leqslant c = c(j; T). \tag{2.6}$$

*Proof.* The method of characteristic describes the solution of (2.1) in the following implicit form

$$\phi(x, t) = h(x - t\phi(x, t)). \tag{2.7}$$

Using the notation

$$y = x - t\phi(x, t), \qquad \text{i.e.} \quad x = y + t\phi(x, t), \tag{2.8}$$

it follows that for $x \neq 0$

$$\frac{\phi(x, t)}{x} = \frac{h(x - t\phi(x, t))}{x} = \frac{h(y)}{y + t\phi(x, t)} = \frac{h(y)}{y + th(y)}. \tag{2.9}$$

Thus, from our assumptions on $h(\cdot)$ (nondecreasing with $h(0) = 0$), (2.9) allows us to conclude that for $x \neq 0$ and $t > 0$

$$\left| \frac{\phi(x, t)}{x} \right| \leqslant \frac{1}{t}, \tag{2.10}$$

which proves (2.4).

To obtain (2.5) we differentiate (2.7) to find that

$$\partial_x \phi(x, t) = \frac{h'(x - t\phi(x, t))}{1 + th'(x - t\phi(x, t))} \geqslant 0. \tag{2.11}$$

Another differentiation of (2.7) gives (after some simplifications)

$$\partial_x^2 \phi(x, t) = \frac{h''(x - t\phi(x, t))}{(1 + th'(x - t\phi(x, t)))^3}. \tag{2.12}$$

To estimate $\|\partial_x^2 \phi(\cdot, t)\|_{L^2}$ we use the change of variable

$$x_0 = x - t\phi(x, t), \qquad dx_0 = (1 - t\partial_x \phi(x, t))\, dx. \tag{2.13}$$

Since

$$(1 - t\partial_x \phi(x, t))(1 + th'(x - t\phi(x, t))) = 1, \tag{2.14}$$

it follows that

$$dx_0 = \frac{dx}{1 + th'(x - t\phi(x, t))}. \tag{2.15}$$

Therefore, using that outside the set $[-M, M]$, $h(x) \sim x^k$ one gets

$$\begin{aligned}
\|\partial_x^2 \phi(\cdot, t)\|_{L^2}^2 &= \int_{-\infty}^{\infty} \left( \frac{h''(x - t\phi(x, t))}{(1 + th'(x - t\phi(x, t)))^3} \right)^2 dx \\
&= \int_{-\infty}^{\infty} \frac{(h''(x_0))^2}{(1 + th'(x_0))^5} \, dx_0 \\
&\leqslant c_k(1+t) + c \int_{|x_0| \geqslant M} \frac{x_0^{2(k-2)}}{(1 + tx_0^{k-1})^5} \, dx_0 \\
&\leqslant c_k(1+t) + \frac{c}{t^{(2k-3)/(k-1)}} \int_{-\infty}^{\infty} \frac{z^{2(k-2)}}{(1 + z^{k-1})^5} \, dz \\
&\leqslant c_k(1+t) + \frac{c}{t^{(2k-3)/(k-1)}}. \qquad (2.16)
\end{aligned}$$

Hence

$$\int_0^T \|\partial_x^2 \phi(\cdot, t)\|_{L^2} \, dt \leqslant c_k(1 + T^2) + cT^{1/2(k-1)} \leqslant c_T, \qquad (2.17)$$

which yields (2.6) with $j = 2$.

A similar computation shows that for $j = 3, 4....$

$$\int_0^T \|\partial_x^j \phi(\cdot, t)\|_{L^2} \, dt \leqslant c(j; T). \qquad (2.18)$$

To establish (2.6) with $j = 3$ we use the formulae (2.7) to deduce that

$$\partial_x^3 \phi(x, t) = \frac{h^{(3)}(x - t\phi(x, t))}{(1 + th'(x - t\phi(x, t)))^4} - \frac{3t(h''(x - t\phi(x, t)))^2}{(1 + th'(x - t\phi(x, t)))^5}. \qquad (2.19)$$

From (2.13), and (2.19) one finds that

$$\begin{aligned}
x\partial_x^3 \phi(x, t) = (x_0 + th(x_0)) \Bigg( &\frac{h^{(3)}(x - t\phi(x, t))}{(1 + th'(x - t\phi(x, t)))^4} \\
&- \frac{3t(h''(x - t\phi(x, t)))^2}{(1 + th'(x - t\phi(x, t)))^5} \Bigg), \qquad (2.20)
\end{aligned}$$

which combined with (2.2) and (2.15) gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |x\partial_x^3\phi(x,t)|^2\,dx \\
&\quad\leqslant 2\int_{-\infty}^{\infty} \frac{(x_0 h^{(3)}(x_0))^2}{(1+th'(x_0))^7}\,dx_0 + 6t^2\int_{-\infty}^{\infty} \frac{x_0^2(h^{(3)}(x_0))^4}{(1+th'(x_0))^9}\,dx_0 \\
&\qquad + 2t^2\int_{-\infty}^{\infty} \frac{(h(x_0)\,h^{(3)}(x_0))^2}{(1+th'(x_0))^7}\,dx_0 + 6t^2\int_{-\infty}^{\infty} \frac{(h(x_0))^2\,(h''(x_0))^4}{(1+th'(x_0))^9}\,dx_0 \\
&\quad\leqslant c(1+t^4) + c\int_{-\infty}^{\infty} \frac{x_0^{2+2(k-3)}}{(1+tx_0^{k-1})^7}\,dx_0 + ct^2\int_{-\infty}^{\infty} \frac{x_0^{4(k-2)+2}}{(1+tx_0^{k-1})^9}\,dx_0 \\
&\qquad + ct^2\int_{-\infty}^{\infty} \frac{x_0^{2(2k-3)}}{(1+tx_0^{k-1})^7}\,dx_0 + ct^4\int_{-\infty}^{\infty} \frac{x_0^{2k+4(k-2)}}{(1+tx_0^{k-1})^9}\,dx_0 \\
&\quad= c(1+t^4) + ct^{-(2k-3)/(k-1)}\int_{-\infty}^{\infty} \frac{z^{2k-4}}{(1+z^{k-1})^7}\,dz \\
&\qquad + ct^{2-(4k-5)/(k-1)}\int_{-\infty}^{\infty} \frac{z^{4k-6}}{(1+z^{k-1})^9}\,dz \\
&\qquad + ct^{4-(6k-7)/(k-1)}\int_{-\infty}^{\infty} \frac{z^{6k-8}}{(1+z^{k-1})^9}\,dz \\
&\quad\leqslant c(1+t^4) + ct^{-(2k-3)/(k-1)}. \qquad (2.21)
\end{aligned}
$$

Therefore

$$\|x\partial_x^3\phi(\cdot,t)\|_{L^2} \leqslant c(1+t^2) + ct^{-(k-3/2)/(k-1)}, \qquad (2.22)$$

which implies that

$$\int_0^T \|x\partial_x^3\phi(\cdot,t)\|_{L^2}\,dt \leqslant (1+T^3). \qquad (2.23)$$

This combined with (2.18) yields (2.6) with $j=3$.

The proof of (2.6) for the higher derivatives $j=4, 5...$ is similar, and it will be omitted.

## 3. LOCAL EXISTENCE OF SOLUTIONS TO THE IVP (1.16)

This section is devoted to the local existence theory for the IVP

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(\phi w) + \partial_x^3 \phi = 0, & t > 0, \quad x \in \mathbb{R}, \\ w(x, 0) = w_0(x), \end{cases} \tag{3.1}$$

where $\phi = \phi(x, t)$ denotes the global solution of the IVP (2.1) described in Theorem 2.1, and $w_0(x) = u_0(x) - h(x) \in \mathcal{S}(\mathbb{R})$.

Our main result in this section can be gathered in the following statement.

THEOREM 3.1. *For any* $w_0 \in \mathcal{S}(\mathbb{R})$ *there exist* $T_0 = T_0(\|w_0\|_{H^5}; \sum_{j=3}^{8} \|\partial_x^j \phi\|_{L_T^1 L_x^2}) > 0$ *and a unique local solution* $w(x, t)$ *of the IVP* (3.1) *which satisfies*

$$w \in \bigcap_{j=1}^{\infty} L^\infty([0, T_0]: H^j(\mathbb{R})) \cap C(\mathbb{R} \times [0, T_0]) \cap C^\infty(\mathbb{R} \times (0, T_0)), \tag{3.2}$$

$$x\partial_x^j w \in L^\infty([0, T_0]: L^2(\mathbb{R})), \qquad j = 0, 1, ...., \tag{3.3}$$

*and*

$$\int_0^{T_0} \int_{-\infty}^{\infty} \partial_x \phi (\partial_x^j w)^2 \, (x, t) \, dx \, dt < \infty, \qquad j = 1, 2, ... \tag{3.4}$$

Our choice of $H^5$, and consequently $8 = 5 + 3$, is used to simplify the presentation of the proof. It guarantees that the solutions are classical, and still allows us to apply a necessary compactness argument. Also we remark that once the result with this index has been established the proof for a larger index reduces to a linear problem.

*Proof.* First, for $N = 1, 2...$ we consider the following IVP

$$\begin{cases} \partial_t w_N + \partial_x^3 w_N + w_N \partial_x w_N + \partial_x(\phi_N w_N) + \partial_x^3 \phi_N = 0, & t > 0, \quad x \in \mathbb{R}, \\ w_N(x, 0) = w_0(x), \end{cases} \tag{3.5}$$

where $\phi_N \in C(\mathbb{R} \times [0, T])$ such that

$$\phi_N(x, t) = \begin{cases} \phi(x, t), & \text{if} \quad |\phi(x, t)| \leqslant N, \\ 2N, & \text{if} \quad \phi(x, t) \geqslant 2N, \\ -2N, & \text{if} \quad \phi(x, t) \leqslant -2N, \end{cases} \tag{3.6}$$

for each $t \in [0, T]$, $\phi(\cdot, t)$ is nondecreasing, and for each $j = 0, 1, \ldots$

$$\sum_{l=0}^{j} |\partial_x^l \phi_N(x, t)| \leqslant c_j \sum_{l=0}^{j} |\partial_x^l \phi(x, t)|. \tag{3.7}$$

Such $\phi_N(x, t)$ can be obtained by $\phi_N(x, t) = \varphi_N(\phi(x, t))$, where $\varphi_N \in C^\infty(\mathbb{R})$, $\varphi_N(\cdot)$ odd, nondecreasing, with $\varphi_N(x) = x$, if $x \in [0, N]$, and $\varphi_N(x) = 2N$, if $x \geqslant 2N$.

The IVP (3.5) has bounded smooth coefficients $\phi_N(\cdot)$, $\partial_x \phi_N(\cdot)$, and smooth external force $\partial_x^3 \phi_N \in L^1([0, T]: H^s(\mathbb{R}))$, for any $s \geqslant 0$, $T > 0$. Hence the artificial viscosity method provides a unique solution $w_N$ of the IVP (3.5) in the time interval $[0, T_N]$, $T_N = T(\|w_0\|_{H^5}; \sum_{j=3}^{8} \|\partial_x^j \phi\|_{L_T^1 L_x^2}; N) > 0$, satisfying

$$w_N \in \bigcap_{j=1}^{\infty} C([0, T_N]: H^j(\mathbb{R})) \cap C^\infty(\mathbb{R} \times [0, T_N]). \tag{3.8}$$

Moreover, as far as the solution $w_N(\cdot)$ belongs to the class described in (3.8) we can perform the following energy estimates.

Multiplying the equation in (3.5) by $w_N$ and integrating the result in the $x$-variable we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} w_N^2 \, dx + \int_{-\infty}^{\infty} \partial_x^3 w_N w_N \, dx + \int_{-\infty}^{\infty} w_N \partial_x w_N w_N \\ + \int_{-\infty}^{\infty} \partial_x(\phi_N w_N) \, w_N \, dx + \int_{-\infty}^{\infty} \partial_x^3 \phi w_N \, dx = 0. \end{aligned} \tag{3.9}$$

By using integration by parts, the second and third terms in (3.9) are zero. To estimate the fourth one we write

$$\int_{-\infty}^{\infty} \partial_x(\phi_N w_N) \, w_N \, dx = \int_{-\infty}^{\infty} \partial_x \phi_N w_N w_N \, dx + \int_{-\infty}^{\infty} \phi_N w_N \partial_x w_N \, dx, \tag{3.10}$$

and for any $a > 0$

$$\begin{aligned} \int_{-a}^{a} \phi_N w_N \partial_x w_N \, dx &= \tfrac{1}{2} \int_{-a}^{a} \phi_N \partial_x (w_N)^2 \, dx \\ &= \tfrac{1}{2} \phi_N (w_N)^2 \big|_{-a}^{a} - \tfrac{1}{2} \int_{-a}^{a} \partial_x \phi_N (w_N)^2 \, dx \\ &\geqslant -\tfrac{1}{2} \int_{-a}^{a} \partial_x \phi_N (w_N)^2 \, dx. \end{aligned} \tag{3.11}$$

Then from (3.10)–(3.11) it follows that

$$\int_{-\infty}^{\infty} \partial_x(\phi_N w_N)\, w_N\, dx \geqslant \tfrac{1}{2} \int_{-\infty}^{\infty} \partial_x \phi_N (w_N)^2\, dx \geqslant 0. \tag{3.12}$$

Collecting the information in (3.9)-(3.12) one finds that

$$\frac{d}{dt} \|w_N(t)\|_{L^2}^2 + \int_{-\infty}^{\infty} \partial_x \phi_N (w_N)^2\, dx \leqslant c\, \|\partial_x^3 \phi(t)\|_{L^2}\, \|w_N(t)\|_{L^2}, \tag{3.13}$$

which combined with (2.6) allows us to obtain the following *a priori* estimates of the solutions of (3.5)

$$\begin{aligned} \sup_{[0,T]} \|w_N(t)\|_{L^2} &\leqslant \|w_0\|_{L^2} + c \int_0^T \|\partial_x^3 \phi(t)\|_{L^2}\, dt \\ &\leqslant c(\|w_0\|_{L^2};\, \|\partial_x^3 \phi\|_{L_T^1 L_x^2}) \equiv c_{0T}, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} \partial_x \phi_N (w_N)^2\, dx\, dt &\leqslant (\|w_0\|_{L^2} + c \int_0^T \|\partial_x^3 \phi(t)\|_{L^2}\, dt)^2 \\ &\leqslant c(\|w_0\|_{L^2};\, \|\partial_x^3 \phi\|_{L_T^1 L_x^2}) \end{aligned} \tag{3.15}$$

where $c$ is independent of $N$.

To obtain estimates of higher derivatives of $w_N$ independent of $N$ we apply $\partial_x^5$ to the equation in (3.5), multiply it by $\partial_x^5 w_N$ and integrate to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} (\partial_x^5 w_N)^2\, dx + \int_{-\infty}^{\infty} \partial_x^8 w_N \partial_x^5 w_N\, dx + \int_{-\infty}^{\infty} \partial_x^5 (w_N \partial_x w_N)\, \partial_x^5 w_N\, dx \\ &\quad + \int_{-\infty}^{\infty} \partial_x^6 (\phi_N w_N)\, \partial_x^5 w_N\, dx + \int_{-\infty}^{\infty} \partial_x^8 \phi \partial_x^5 w_N\, dx = 0. \end{aligned} \tag{3.16}$$

As in the previous case, integration by parts shows that the second term above is zero. Also a standard argument (see [3]) shows that

$$\left| \int_{-\infty}^{\infty} \partial_x^5 (w_N \partial_x w_N)\, \partial_x^5 w_N\, dx \right| \leqslant c\, \|\partial_x w_N(t)\|_{L^\infty}\, \|\partial_x^5 w_N(t)\|_{L^2}^2, \tag{3.17}$$

and

$$\left| \int_{-\infty}^{\infty} \partial_x^8 \phi \partial_x^5 w_N\, dx \right| \leqslant \|\partial_x^8 \phi(t)\|_{L^2}\, \|\partial_x^5 w_N(t)\|_{L^2}. \tag{3.18}$$

To handle the fourth term in (3.16) we write

$$\int_{-\infty}^{\infty} \partial_x^6(\phi_N w_N)\, \partial_x^5 w_N\, dx$$

$$= \int_{-\infty}^{\infty} \phi_N \partial_x^6 w_N \partial_x^5 w_N\, dx + 6 \int_{-\infty}^{\infty} \partial_x \phi_N (\partial_x^5 w_N)^2\, dx + \Phi. \tag{3.19}$$

The argument used in (3.10)–(3.12) shows that

$$\int_{-\infty}^{\infty} \phi_N \partial_x^6 w_N \partial_x^5 w_N\, dx + 6 \int_{-\infty}^{\infty} \partial_x \phi_N (\partial_x^5 w_N)^2\, dx$$

$$\geqslant \tfrac{11}{2} \int_{-\infty}^{\infty} \partial_x \phi_N (\partial_x^5 w_N)^2\, dx \geqslant 0, \tag{3.20}$$

and Sobolev lemma, and (3.14) give

$$\begin{aligned}
|\Phi| &\leqslant \|(\partial_x^6(\phi_N w_N) - \phi_N \partial_x^6 w_N - 6\partial_x \phi_N \partial_x^5 w_N)(t)\|_{L^2}\, \|\partial_x^5 w_N(t)\|_{L^2} \\
&\leqslant c \left( \sum_{j=2}^{6} \|\partial_x^j \phi_N(t)\|_{L^2}\, \|\partial_x^{6-j} w_N(t)\|_{L^\infty} \right) \|\partial_x^5 w_N(t)\|_{L^2} \\
&\leqslant c \sum_{j=2}^{6} \|\partial_x^j \phi_N(t)\|_{L^2}\, [\, \|w_N(t)\|_{L^2} + \|\partial_x^5 w_N(t)\|_{L^2}]\, \|\partial_x^5 w_N(t)\|_{L^2} \\
&\leqslant c \sum_{j=2}^{6} \|\partial_x^j \phi_N(t)\|_{L^2}\, [c_{0T} + \|\partial_x^5 w_N(t)\|_{L^2}]\, \|\partial_x^5 w_N(t)\|_{L^2}.
\end{aligned} \tag{3.21}$$

Gathering the information in (3.16)–(3.21) we find that for $t \in [0, T]$

$$\begin{aligned}
&\frac{d}{dt} \|\partial_x^5 w_N(t)\|_{L^2}^2 + 11 \int_{-\infty}^{\infty} \partial_x \phi_N (\partial_x^5 w_N)^2\, dx \\
&\qquad \leqslant c\, \|w_N(t)\|_{L^\infty}\, \|\partial_x^5 w_N(t)\|_{L^2}^2 \\
&\qquad\quad + c \sum_{j=2}^{6} \|\partial_x^j \phi_N(t)\|_{L^2}\, [c_{0T} + \|\partial_x^5 w_N(t)\|_{L^2}]\, \|\partial_x^5 w_N(t)\|_{L^2} \\
&\qquad\quad + \|\partial_x^8 \phi(t)\|_{L^2}\, \|\partial_x^5 w_N(t)\|_{L^2} \\
&\qquad \leqslant c[c_{0T} + \|\partial_x^5 w_N(t)\|_{L^2}]\, \|\partial_x^5 w_N(t)\|_{L^2}^2 \\
&\qquad\quad + c \sum_{j=2}^{6} \|\partial_x^j \phi_N(t)\|_{L^2}\, [c_{0T} + \|\partial_x^5 w_N(t)\|_{L^2}]\, \|\partial_x^5 w_N(t)\|_{L^2} \\
&\qquad\quad + \|\partial_x^8 \phi(t)\|_{L^2}\, \|\partial_x^5 w_N(t)\|_{L^2},
\end{aligned} \tag{3.22}$$

which allows us to conclude that there exists $T_0 = T_0(\|w_0\|_{H^5}; \sum_{j=3}^{8} \|\partial_x^j \phi\|_{L_T^1 L_x^2}) > 0$, such that

$$\sup_{[0,\, T_0]} \|w_N(t)\|_{H^5} \leqslant c = c\left(\|w_0\|_{H^5}; \sum_{j=3}^{8} \|\partial_x^j \phi\|_{L_T^1 L_x^2}\right) \tag{3.23}$$

and

$$\int_0^{T_0} \int_{-\infty}^{\infty} \partial_x \phi_N (\partial_x^5 w_N)^2 \, dx\, dt \leqslant c = c\left(\|w_0\|_{H^5}; \sum_{j=3}^{8} \|\partial_x^j \phi\|_{L_T^1 L_x^2}\right), \tag{3.24}$$

where the constants in the right hand side of (3.23)–(3.24) do not dependent on $N$.

Once the *a priori* estimate (3.23) has been established it is easy (basically a linear problem, see [3]) to extend it to $H^s(\mathbb{R})$, $s \geqslant 5$.

Also, the inequality (3.23) allows us to extend the local solution $w_N$ of the IVP (3.5) defined in $[0, T_N]$ to the whole time interval $[0, T_0]$, for $N = 1, 2, \ldots$.

Hence, using the equation in (3.5) and the estimates in Theorem 2.1 one has that

$$\{w_N\}_{N=1}^{\infty} \subseteq \bigcap_{j=1}^{\infty} C([0, T_0] \colon H^j(\mathbb{R})), \tag{3.25}$$

and for $l = 0, 1, \ldots$

$$\{\partial_t^l w_N\}_{N=1}^{\infty} \subseteq C(\mathbb{R} \times [0, T_0]) \cap L^{\infty}_{\text{loc}}(\mathbb{R} \times [0, T_0]), \tag{3.26}$$

with

$$\sup_{|x| \leqslant R,\, 0 \leqslant t \leqslant T_0} |\partial_t^n w_N(x, t)| \leqslant c\left(n; R; T_0; \|w_0\|_{H^{3n}}; \sum_{j=3}^{3n} \|\partial_x^j \phi\|_{L_T^1 L_x^2}\right). \tag{3.27}$$

Hence, there exists a subsequence $\{w_{N_l}\}_{l=1}^{\infty}$ such that

$$\partial_t^{\alpha_1} \partial_x^{\alpha_2} w_{N_l} \to \partial_t^{\alpha_1} \partial_x^{\alpha_2} w, \qquad |\alpha_1| + |\alpha_2| \leqslant 4, \tag{3.28}$$

uniformly on any compact set of $\mathbb{R} \times [0, T_0]$. Thus, it also follows that

$$w \in \bigcap_{j=1}^{\infty} L^{\infty}([0, T_0] \colon H^j(\mathbb{R})) \cap C^{\infty}(\mathbb{R} \times [0, T_0]), \tag{3.29}$$

with

$$\partial_t w + \partial_x^3 w + w \partial_x w + \partial_x(\phi w) + \partial_x^3 \phi = 0, \qquad t \in [0, T_0], \quad x \in \mathbb{R}, \tag{3.30}$$

$$\int_0^{T_0}\int_{-\infty}^{\infty} \partial_x\phi(\partial_x^j w)^2\,dx\,dt \leqslant c = c\left(\|w_0\|_{H^j};\ \sum_{l=3}^{j+3}\|\partial_x^l\phi\|_{L_T^1L_x^2}\right),$$

$$j=1, 2, \ldots \tag{3.31}$$

and

$$w(x,0)=w_0(x). \tag{3.32}$$

Next we show that the solution $w(x,t)$ of the IVP (3.1) satisfies the weighted estimates (3.3) as long as it belongs to the class described in (3.29). As in the previous argument we need to handle the integration by parts carefully.

First, define $\psi_n \in C^\infty(\mathbb{R})$, $\psi_n$ even, nonnegative, with $0 \leqslant \psi_n'(x) \leqslant 1$, $\psi_n(x)-n \geqslant (x-n)\,\psi_n'(x)$, if $n \leqslant x \leqslant n+1$, (this follows by assuming that $\psi_n''(x) \leqslant 0$, if $n \leqslant x \leqslant n+1$) and $\psi_n(x)=x$, if $x\in[1,n]$, $\psi_n(x)=n+1/2$, if $x \geqslant n+1$.

Multiplying the equation (3.1) by $w\psi_n$ and integrating the result we find, after using integration by parts, that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2\psi_n\,dx + \frac{3}{2}\int_{-\infty}^{\infty}(\partial_x w)^2\,\psi_n'\,dx - \frac{1}{2}\int_{-\infty}^{\infty} w^2\psi_n^{(3)}\,dx$$

$$+\int_{-\infty}^{\infty}\partial_x w w^2\psi_n\,dx + \int_{-\infty}^{\infty}\partial_x(\phi w)\,w\psi_n\,dx + \int_{-\infty}^{\infty}\partial_x^3\phi w\psi_n\,dx = 0. \tag{3.33}$$

A direct argument provides the following bounds

$$\left|\int_{-\infty}^{\infty}(\partial_x w)^2\,\psi_n'\,dx\right| \leqslant \|\partial_x w(t)\|_{L^2}^2, \tag{3.34}$$

$$\left|\int_{-\infty}^{\infty} w^2\psi_n^{(3)}\,dx\right| \leqslant c\,\|w(t)\|_{L^2}^2, \tag{3.35}$$

$$\left|\int_{-\infty}^{\infty}\partial_x w w^2\psi_n\,dx\right| \leqslant c\,\|\partial_x w(t)\|_{L^\infty}\,\|w(t)\sqrt{\psi_n}\|_{L^2}^2, \tag{3.36}$$

and

$$\left|\int_{-\infty}^{\infty}\partial_x^3\phi w\psi_n\,dx\right| \leqslant \|\partial_x^3\phi(t)\sqrt{\psi_n}\|_{L^2}\,\|w(t)\sqrt{\psi_n}\|_{L^2}, \tag{3.37}$$

where the constants in (3.34)–(3.37) are independent of $n$.

To bound the fifth term in (3.33) we use an argument similar to that given in (3.10)–(3.12) to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \partial_x(\phi w)\, w\psi_n\, dx \\
&\quad = \int_{-\infty}^{\infty} \partial_x \phi w^2 \psi_n\, dx + \int_{-\infty}^{\infty} \phi w \partial_x w \psi_n\, dx \\
&\quad \geqslant \int_{-\infty}^{\infty} \partial_x \phi w^2 \psi_n\, dx - \tfrac{1}{2} \int_{-\infty}^{\infty} \partial_x \phi w^2 \psi_n\, dx - \tfrac{1}{2} \int_{-\infty}^{\infty} \phi w^2 \psi'_n\, dx \\
&\quad \geqslant \tfrac{1}{2} \int_{-\infty}^{\infty} (\partial_x \phi \psi_n - \phi \psi'_n)\, w^2\, dx \geqslant -c_0 \int_{-\infty}^{\infty} w^2\, dx. \qquad (3.38)
\end{aligned}$$

In the last inequality we have used that

$$F(x, t) \equiv \partial_x \phi(x, t)\, \psi_n(x) - \phi(x, t)\, \psi'_n(x) \geqslant -c_0, \qquad (3.39)$$

which we shall prove next. First, we remark that a direct calculation shows that the data $h(\cdot)$ of the IVP (2.1) described in (1.8), (or more general (2.2)) satisfies the following:

there exist constants $c_0, c_1 \geqslant 0$ such that for any $x \in \mathbb{R}$

$$|x|\, h'(x) \geqslant |h(x)| - c_0, \qquad (3.40)$$

and

$$|x|\, h'(x) \geqslant 2\, |h(x)| - c_1\, |x| - c_0. \qquad (3.41)$$

For $|x| \geqslant n+1$, $\psi_n(x) = n + 1/2$, and we have $F(x, t) = \partial_x \phi \psi_n \geqslant 0$.

For $1 \leqslant x \leqslant n$, $\psi(x) = x$, and $\psi'_n = 1$, then from (2.13), and (3.40) one sees that

$$\begin{aligned}
F(x, t) &= \frac{x h'(x - t\phi(x, t))}{1 + th'(x - t\phi(x, t))} - h(x - t\phi(x, t))\, \frac{1 + th'(x - t\phi(x, t))}{1 + th'(x - t\phi(x, t))} \\
&= \frac{(x_0 + th(x_0))\, h'(x_0)}{1 + th'(x_0)} - h(x_0)\, \frac{1 + th'(x_0)}{1 + th'(x_0)} = \frac{x_0 h'(x_0) - h(x_0)}{1 + th'(x_0)} \\
&\geqslant \frac{-c_0}{1 + th'(x_0)} \geqslant -c_0. \qquad (3.42)
\end{aligned}$$

A similar argument works for $-n \geqslant x \geqslant -1$.

For $n \leqslant x \leqslant n+1$ we have

$$\begin{aligned} F(x,t) &= \partial_x \phi(\cdot)\,\psi_n - \phi(\cdot)\,\psi'_n \geqslant \partial_x \phi(\cdot)((x-n)\,\psi'_n + n) - \phi(\cdot)\,\psi'_n \\ &= x\partial_x\phi(\cdot)\,\psi'_n(x) - n\partial_x\phi(\cdot)\,\psi'_n + n\partial_x\phi(\cdot) - \phi(\cdot)\,\psi'_n(x) \\ &= (x\partial_x\phi(\cdot) - \phi(\cdot))\,\psi'_n + (n - n\psi'_n)\,\partial_x\phi(\cdot) \geqslant 0 \end{aligned} \tag{3.43}$$

since (3.40) and our hypothesis on $\psi_n$ show that both terms in the last line in (3.43) are nonnegative.

The case $-(n+1) \leqslant x \leqslant -n$ is similar to the previous one.

Thus we have established (3.39) and completed the proof of (3.38). Collecting the results in (3.33)–(3.38) we have that for $t \in (0, T_0)$

$$\begin{aligned} \frac{d}{dt}\|w\sqrt{\psi_n}\,(t)\|^2_{L^2} &\leqslant \|\partial_x w(t)\|_{L^\infty}\,\|w\sqrt{\psi_n}\,(t)\|^2_{L^2} \\ &\quad + \|\partial_x^3\phi\sqrt{|x|}\|_{L^2}\,\|w\sqrt{\psi_n}\,(t)\|_{L^2} + \|w(t)\|_{H^1}, \end{aligned} \tag{3.44}$$

which combined with (3.29) shows that

$$\sup_{[0,T_0]}\|w\sqrt{\psi_n}\,(t)\|_{L^2} \leqslant c = c(\|w_0\sqrt{|x|}\|_{L^2};\ \|w_0\|_{H^5};\ \|\partial_x^3\phi\sqrt{|x|}\|_{L^1{}_{T_0}L^2_x}), \tag{3.45}$$

where the constant $c$ is independent of $n$. Therefore by taking $n \uparrow \infty$ we obtain that

$$\sup_{[0,T_0]}\|w(t)\sqrt{|x|}\|_{L^2} \leqslant c. \tag{3.46}$$

In the same manner one has for $j = 0, 1, \ldots$ that

$$\begin{aligned} &\sup_{[0,T_0]}\|\partial_x^j w(t)\sqrt{|x|}\|_{L^2} \\ &\leqslant c = c\left(\|\partial_x^j w_0\sqrt{|x|}\|_{L^2};\ \|w_0\sqrt{|x|}\|_{L^2};\ \|w_0\|_{H^{j+5}};\ \sum_{l=3}^{j+3}\|\partial_x^3\phi\sqrt{|x|}\|_{L^1{}_{T_0}L^2_x}\right). \end{aligned} \tag{3.47}$$

Once the estimate (3.47) is available it can be used to obtain stronger weighted estimates. These will be established by a similar argument.

Multiplying the equation in (3.1) by $w\psi_n^2$ and integrating the result we find, after using integration by parts, that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2\psi_n^2\,dx+3\int_{-\infty}^{\infty}(\partial_x w)^2\,\psi_n'\psi_n\,dx-\frac{1}{2}\int_{-\infty}^{\infty} w^2(\psi_n^2)^{(3)}\,dx$$
$$+\int_{-\infty}^{\infty}\partial_x w w^2\psi_n^2\,dx+\int_{-\infty}^{\infty}\partial_x(\phi w)\,w\psi_n^2\,dx+\int_{-\infty}^{\infty}\partial_x^3\phi w\psi_n^2\,dx=0. \tag{3.48}$$

A direct argument and (3.47) yield the bounds

$$\left|\int_{-\infty}^{\infty}(\partial_x w)^2\,\psi_n'\psi_n\,dx\right|\leqslant c\,\|\partial_x w(t)\sqrt{|x|}\|_{L^2}^2)\leqslant c, \tag{3.49}$$

$$\left|\int_{-\infty}^{\infty} w^2(\psi_n^2)^{(3)}\,dx\right|\leqslant c\,\|w(t)\|_{L^2}^2, \tag{3.50}$$

$$\left|\int_{-\infty}^{\infty}\partial_x w w^2\psi_n^2\,dx\right|\leqslant c\,\|\partial_x w(t)\|_{L^\infty}\,\|w(t)\,\psi_n\|_{L^2}^2\leqslant c, \tag{3.51}$$

and

$$\left|\int_{-\infty}^{\infty}\partial_x^3\phi w\psi_n^2\,dx\right|\leqslant\|x\partial_x^3\phi(t)\|_{L^2}\,\|w(t)\,\psi_n\|_{L^2}, \tag{3.52}$$

where the constants in (3.49)–(3.52) are independent of $n$.

To bound the fifth term in (3.48) we combine the inequalities

$$\begin{aligned}\int_{-\infty}^{\infty}\partial_x(\phi w)\,w\psi_n^2\,dx&=\int_{-\infty}^{\infty}\partial_x\phi w^2\psi_n^2\,dx+\int_{-\infty}^{\infty}\phi w\partial_x w\psi_n^2\,dx\\&\geqslant\tfrac{1}{2}\int_{-\infty}^{\infty}(\partial_x\phi\psi_n^2-2\phi\psi_n\psi_n')\,w^2\,dx\\&\geqslant-c_1\int_{-\infty}^{\infty}w^2\psi_n^2\,dx-c_0\int_{-\infty}^{\infty}w^2\,dx,\end{aligned} \tag{3.53}$$

where in the last step we have used (3.41) and an argument similar to that described in (3.42)–(3.43) to see that

$$G(x,t)=\partial_x\phi\psi_n^2-2\phi\psi_n\psi_n'\geqslant-c_1\psi_n^2-c_0. \tag{3.54}$$

From (3.48)–(3.54) it follows that for $t \in (0, T_0)$

$$\frac{d}{dt}\|w\psi_n(t)\|_{L^2}^2 \leqslant c(1+\|\partial_x w(t)\|_{L^\infty})\,\|w\psi_n(t)\|_{L^2}^2 + \|\partial_x w(t)\sqrt{|x|}\|_{L^2}^2 + \|x\partial_x^3\phi(t)\|_{L^2}\,\|w\psi_n(t)\|_{L^2} + \|w(t)\|_{H^1}^2, \tag{3.55}$$

which combined with (3.29), and (3.47) shows that

$$\sup_{[0,T_0]}\|w\psi_n(t)\|_{L^2} \leqslant c = c\left(\|xw_0\|_{L^2};\ \|w_0\|_{H^5};\ \sum_{j=3}^{4}\|(1+|x|)\,\partial_x^j\phi\|_{L^1_{T_0}L^2_x}\right), \tag{3.56}$$

with $c$ independent of $n$. Letting $n \uparrow \infty$ we obtain that

$$\sup_{[0,T_0]}\|xw(t)\|_{L^2} \leqslant c = c\left(\|xw_0\|_{L^2};\ \|w_0\|_{H^5};\ \sum_{j=3}^{4}\|(1+|x|)\,\partial_x^j\phi\|_{L^1_{T_0}L^2_x}\right). \tag{3.57}$$

Using the same argument one gets for that $j = 0, 1, \ldots$

$$\sup_{[0,T_0]}\|x\partial_x^j w(t)\|_{L^2} \leqslant c = c\left(\|x\partial_x^j w_0\|_{L^2};\ \|xw_0\|_{L^2};\ \|w_0\|_{H^{j+5}};\ \sum_{l=3}^{j+4}\|(1+|x|)\,\partial_x^l\phi\|_{L^1_{T_0}L^2_x}\right). \tag{3.58}$$

Combining the estimates (2.3)–(2.6), (3.58), and (3.29) it follows that for any $j = 1, 2, \ldots$, $\|\partial_t^j w(t)\|_{L^2} \in C((0, T_0])$.

Finally we prove the uniqueness of the solution of the IVP (3.1). Assuming that $w_1$, $w_2$ are solutions of (3.1) in the class described in (3.29), then $v = w_1 - w_2$ solves the IVP

$$\begin{cases} \partial_t v + \partial_x^3 v + w_1\partial_x v + \partial_x w_2 v + \partial_x(\phi v) = 0 & t \in (0, T_0), \quad x \in \mathbb{R} \\ v(x, 0) = 0. \end{cases} \tag{3.60}$$

Using that (for details see (3.10)–(3.12))

$$\int_{-\infty}^{\infty} \partial_x(\phi v)\, v\, dx \geqslant 0, \tag{3.61}$$

one easily sees that

$$\frac{d}{dt}\|v(t)\|_{L^2}\leqslant c(\|\partial_x w_1(t)\|_{L^\infty}+\|\partial_x w_2(t)\|_{L^\infty})\|v(t)\|_{L^2}, \tag{3.62}$$

which yields the desired result.

## 4. PROOF OF THEOREM 1.2

First we recall the inequalities (3.13), and the first one in (3.22) for solutions $w_N$ of the IVP (3.5)

$$\frac{d}{dt}\|w_N(t)\|_{L^2}^2+\int_{-\infty}^{\infty}\partial_x\phi_N(w_N)^2\,dx\leqslant c\,\|\partial_x^3\phi(t)\|_{L^2}\,\|w_N(t)\|_{L^2}, \tag{4.1}$$

and

$$\begin{aligned}&\frac{d}{dt}\|\partial_x^5 w_N(t)\|_{L^2}^2+11\int_{-\infty}^{\infty}\partial_x\phi_N(\partial_x^5 w_N)^2\,dx\\&\leqslant c\,\|w_N(t)\|_{L^\infty}\,\|\partial_x^5 w_N(t)\|_{L^2}^2\\&\quad+c\sum_{j=2}^{6}\|\partial_x^j\phi_N(t)\|_{L^2}\,[c(\|w_0\|_{L^2};\|\partial_x^3\phi\|_{L_T^1L_x^2})+\|\partial_x^5 w_N(t)\|_{L^2}]\,\|\partial_x^5 w_N(t)\|_{L^2}\\&\quad+\|\partial_x^8\phi(t)\|_{L^2}\,\|\partial_x^5 w_N(t)\|_{L^2}.\end{aligned} \tag{4.2}$$

In the proof of (4.1)–(4.2) we only used the fact that $\phi_N$ is nondecreasing. Therefore the same argument applies to the local solution $w(\cdot)$ of the IVP (3.1) provided by Theorem 3.1 with $s=1, 2, .., s+1$, and $s+3$ in (4.2) instead of 5, 6, and 8 respectively. Thus, we rewrite (4.1)–(4.2) as

$$\frac{d}{dt}\|w(t)\|_{L^2}\leqslant c\,\|\partial_x^3\phi(t)\|_{L^2}, \tag{4.3}$$

and for $s=1, 2, ..$

$$\begin{aligned}&\frac{d}{dt}\|\partial_x^s w(t)\|_{L^2}\\&\qquad\leqslant c_s\,\|w(t)\|_{L^\infty}\,\|\partial_x^s w(t)\|_{L^2}\\&\qquad\quad+c_s\sum_{j=2}^{s+1}\|\partial_x^j\phi(t)\|_{L^2}\,[c(\|w_0\|_{L^2};\|\partial_x^3\phi\|_{L_T^1L_x^2})+\|\partial_x^s w_N(t)\|_{L^2}]\\&\qquad\quad+\|\partial_x^{s+3}\phi(t)\|_{L^2}.\end{aligned} \tag{4.4}$$

From (4.4) and Theorem 3.1 it follows that in order to obtain an *a priori* estimate of the local solution $w(\cdot)$ which allows us to extend it to the whole interval $[0, T']$, $T' > T_0$ it suffices to get an *a priori* bound of the form

$$\int_{T_0/2}^{T'} \|w(t)\|_{L^\infty}\, dt \leqslant c, \tag{4.5}$$

with $c$ depending on $T'$ and the estimates in (2.4)–(2.6) in $[T_0/2, T']$ for $\phi$, and $\|w_0\|_5$.

By the Sobolev lemma it suffices to get an *a priori* bound for the quantity

$$\sup_{[T_0/2,\, T']} \|w(t)\|_{H^2} \leqslant c. \tag{4.6}$$

The inequality (4.3) provides an appropriate bound

$$\sup_{[T_0/2,\, T']} \|w(t)\|_{L^2} \leqslant c(\|w_0\|_{L^2}; \|\partial_x^3 \phi(t)\|_{L^1_T L^2}) \equiv c_{0T'}. \tag{4.7}$$

Next we shall get an *a priori* bound for $\sup_{[T_0/2,\, T']} \|\partial_x w(t)\|_{L^2}$. We shall use the conservation laws for the KdV equation deduced in [6]. In this case, due to the extra terms in the equation in (3.1), they just show that our solution does not blow up in finite time. Differentiating the equation in (3.1), multiplying the result by $\partial_x w$, adding this to the product of the equation in (3.1) with $-w^2/2$ and integrating the result we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left((\partial_x w)^2 - \frac{1}{3} w^3\right) dx + \int_{-\infty}^{\infty} (\partial_x^3 w + w\partial_x w)\left(\partial_x^2 w - \frac{1}{2} w^2\right) dx \\
&\quad + \int_{-\infty}^{\infty} \partial_x^2(\phi w)\, \partial_x w\, dx + \int_{-\infty}^{\infty} \partial_x^4 \phi \partial_x w\, dx - \frac{1}{2}\int_{-\infty}^{\infty} \partial_x(\phi w)\, w^2\, dx \\
&\quad - \frac{1}{2}\int_{-\infty}^{\infty} \partial_x^3 \phi w^2\, dx = 0.
\end{aligned} \tag{4.8}$$

As in the case of the KdV equation

$$\int_{-\infty}^{\infty} (\partial_x^3 w + w\partial_x w)(\partial_x^2 w - \tfrac{1}{2} w^2)\, dx = 0. \tag{4.9}$$

Also, the argument in (3.10)–(3.12) gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} \partial_x^2(\phi w)\, \partial_x w\, dx \\
&\quad = \int_{-\infty}^{\infty} \partial_x^2 \phi w \partial_x w\, dx + 2\int_{-\infty}^{\infty} \partial_x \phi (\partial_x w)^2\, dx + \int_{-\infty}^{\infty} \phi \partial_x^2 w \partial_x w\, dx \\
&\quad \geqslant \int_{-\infty}^{\infty} \partial_x^2 \phi w \partial_x w\, dx + \tfrac{3}{2}\int_{-\infty}^{\infty} \partial_x \phi (\partial_x w)^2\, dx \geqslant \int_{-\infty}^{\infty} \partial_x^2 \phi w \partial_x w\, dx,
\end{aligned} \tag{4.10}$$

with

$$\left|\int_{-\infty}^{\infty} \partial_x^2 \phi w \partial_x w \, dx\right| \leqslant c \, \|\partial_x^2 \phi(t)\|_{L^2} \left[\, \|w(t)\|_{L^2} + \|\partial_x w(t)\|_{L^2}\right] \|\partial_x w(t)\|_{L^2}. \tag{4.11}$$

To bound the fifth term in (4.7) we write

$$\begin{aligned} -\tfrac{1}{2}\int_{-\infty}^{\infty} \partial_x(\phi w)\, w^2 \, dx &= -\tfrac{1}{2}\int_{-\infty}^{\infty} \partial_x \phi w^3 \, dx - \tfrac{1}{6}\int_{-\infty}^{\infty} \phi \partial_x(w^3)\, dx \\ &= -\tfrac{1}{3}\int_{-\infty}^{\infty} \partial_x \phi w^3 \, dx, \end{aligned} \tag{4.12}$$

where we used that

$$\int_{-\infty}^{\infty} \phi \partial_x(w^3)\, dx = \phi w^3 \,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \partial_x \phi w^3 \, dx = -\int_{-\infty}^{\infty} \partial_x \phi w^3 \, dx \tag{4.13}$$

since from (2.4), (3.58), and the Sobolev lemma it follows that for $t \geqslant T_0/2$

$$\phi w^3 \,|_{-\infty}^{\infty} = 0. \tag{4.14}$$

From the Gagliardo–Nirenberg inequality, (2.5), and (4.7) we find that

$$\begin{aligned} \left|\int_{-\infty}^{\infty} \partial_x \phi w^3 \, dx\right| &\leqslant c \, \|\partial_x \phi(t)\|_{L^\infty} \, \|w(t)\|_{L^3}^3 \\ &\leqslant \frac{1}{t} \, \|w(t)\|_{L^2}^{5/2} \, \|\partial_x w(t)\|_{L^2}^{1/2} \\ &\leqslant \frac{1}{t} \, c_{0T'} \, \|\partial_x w(t)\|_{L^2}^{1/2} \\ &\leqslant \frac{1}{t} \, c_{0T'} + \frac{1}{t} \, \|\partial_x w(t)\|_{L^2}. \end{aligned} \tag{4.15}$$

In the same manner one gets using again the notation in (4.7) that

$$\begin{aligned} \left|\int_{-\infty}^{\infty} \partial_x^3 \phi w^2 \, dx\right| &\leqslant \|\partial_x^3 \phi(t)\|_{L^2} \, \|w(t)\|_{L^4}^2 \\ &\leqslant c_{0T'} \, \|\partial_x^3 \phi(t)\|_{L^2} + \|\partial_x^3 \phi(t)\|_{L^2} \, \|\partial_x w(t)\|_{L^2}, \end{aligned} \tag{4.16}$$

and

$$\left|\int_{-\infty}^{\infty} \partial_x^4\phi\partial_x w\,dx\right| \leqslant c\,\|\partial_x^4\phi(t)\|_{L^2}\,\|\partial_x w(t)\|_{L^2}. \tag{4.17}$$

Also by the Gagliardo–Nirenberg inequality one finds that

$$\begin{aligned}\Gamma_1^2(t) &\equiv \int_{-\infty}^{\infty} ((\partial_x w)^2 - \tfrac{1}{3}w^3)\,dx \geqslant \|\partial_x w(t)\|_{L^2}^2 - c\,\|\partial_x w(t)\|_{L^2}^{1/2}\,\|w(t)\|_{L^2}^{5/2}\\ &\geqslant \|\partial_x w(t)\|_{L^2}^2 - [\tfrac{1}{2}\|\partial_x w(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^{10/3}]\\ &\geqslant \tfrac{1}{2}\|\partial_x w(t)\|_{L^2}^2 - c\,\|w(t)\|_{L^2}^{10/3},\end{aligned} \tag{4.18}$$

which implies that

$$\|\partial_x w(t)\|_{L^2}^2 \leqslant 2\Gamma_1(t) + c\,\|w(t)\|_{L^2}^{10/3}. \tag{4.19}$$

Collecting the information in (4.9)–(4.19) we rewrite (4.8) as

$$\begin{aligned}\frac{d}{dt}\Gamma_1^2(t) &\leqslant c\,\|\partial_x^3\phi(t)\|_{L^2}\,\Gamma_1^2(t)\\ &\quad + c\left(\frac{1}{t} + \|\partial_x^3\phi(t)\|_{L^2} + \|\partial_x^4\phi(t)\|_{L^2}\right)\Gamma_1(t)\\ &\quad + c\left(\frac{1}{t}\,c_{0T'} + \|\partial_x^3\phi(t)\|_{L^2}\right).\end{aligned} \tag{4.20}$$

Integrating (4.20) in the interval $[T_0/2, T']$ we get an *a priori* estimate for $\sup_{[T_0/2, T']}\Gamma_1(t)$ which combined with (4.7) shows that

$$\sup_{[T_0/2, T']} \|\partial_x w\|_{L^2} \leqslant c\left(\|w_0\|_{H^1};\ \sum_{j=2}^{4}\|\partial_x^j\phi(t)\|_{L^1_{T'}L^2_x}\right). \tag{4.21}$$

Finally we have the estimate for $\partial_x^2 w$ in $L^2(\mathbb{R})$.

Differentiating twice the equation in (3.1), multiplying the result by $2\partial_x^2 w$, adding this to the product of the equation with

$$\Lambda_1(x, t) \equiv \tfrac{10}{3}\,w\partial_x^2 w + \tfrac{5}{3}(\partial_x w)^2 + \tfrac{5}{9}w^3, \tag{4.22}$$

and integrating the result we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left((\partial_x^2 w)^2+\frac{5}{36}w^4-\frac{5}{3}w(\partial_x w)^2\right)dx$$
$$+2\int_{-\infty}^{\infty}\partial_x^2(\partial_x^3 w+w\partial_x w)\,\partial_x^2 w\,dx$$
$$+\int_{-\infty}^{\infty}\partial_x^2(\partial_x^3 w+w\partial_x w)\,\Lambda_1\,dx+2\int_{-\infty}^{\infty}\partial_x^3(\phi w)\,\partial_x^2 w\,dx$$
$$+\int_{-\infty}^{\infty}\partial_x(\phi w)\,\Lambda_1\,dx+2\int_{-\infty}^{\infty}\partial_x^5\phi\partial_x^2 w\,dx+\int_{-\infty}^{\infty}\partial_x^3\phi\Lambda_1\,dx=0. \tag{4.23}$$

In (4.23), as below, we have combined the estimates (2.4)–(2.5), and (3.58), the Sobolev lemma and the fact that $t\geqslant T_0/2>0$ to justify all the integration by parts used.

As in the case of the KdV equation the addition of the second and third terms in (4.23) vanishes.

To estimate the fourth term we write

$$\int_{-\infty}^{\infty}\partial_x^3(\phi w)\,\partial_x^2 w\,dx=\int_{-\infty}^{\infty}\phi\partial_x^3 w\partial_x^2 w\,dx+3\int_{-\infty}^{\infty}\partial_x\phi(\partial_x^2 w)^2\,dx+\Psi_2(t)$$
$$\geqslant\Psi_2(t), \tag{4.24}$$

and a well-known argument leads to

$$|\Psi_2(t)|\leqslant c\sum_{j=2}^{3}\|\partial_x^j\phi(t)\|_{L^2}\left[\|\partial_x^2 w(t)\|_{L^2}+\|w(t)\|_{L^2}\right]\|\partial_x^2 w(t)\|_{L^2}. \tag{4.25}$$

To bound the fifth term in (4.23) we use that

$$\int_{-\infty}^{\infty}\phi\partial_x w w^3\,dx=-\tfrac{1}{4}\int_{-\infty}^{\infty}\partial_x\phi w^4\,dx, \tag{4.26}$$

and

$$\int_{-\infty}^{\infty}\phi((\partial_x w)^3+2w\partial_x w\partial_x^2 w)\,dx=-\int_{-\infty}^{\infty}\partial_x\phi w(\partial_x w)^2\,dx. \tag{4.27}$$

As we mentioned, the estimates (2.4)–(2.5), and (3.58), the Sobolev lemma and the fact that $t\geqslant T_0/2>0$ justify all the integration by parts used.

Thus (4.26)–(4.27) and the following three inequalities provide the appropriate bound for the fifth term in (4.23)

$$\left|\int_{-\infty}^{\infty} \partial_x \phi w^4\, dx\right| \leqslant \frac{c}{t}\, \|w\|_{L^4}^4 \leqslant \frac{c}{t}\, \|\partial_x^2 w\|_{L^2}^{1/2}\, \|w\|_{L^2}^{3/2}, \tag{4.28}$$

$$\left|\int_{-\infty}^{\infty} \partial_x \phi w(\partial_x w)^2\, dx\right| \leqslant \frac{c}{t}\, \|\partial_x^2 w\|_{L^2}^{5/4}\, \|w\|_{L^2}^{7/4}, \tag{4.29}$$

and

$$\left|\int_{-\infty}^{\infty} \partial_x \phi w^2\, \partial_x^2 w\, dx\right| \leqslant \frac{c}{t}\, \|\partial_x^2 w\|_{L^2}^{5/4}\, \|w\|_{L^2}^{7/4}. \tag{4.30}$$

Last we have the bounds for the sixth and seventh terms in (4.23)

$$\left|\int_{-\infty}^{\infty} \partial_x^5 \phi w\, \partial_x^2 w\, dx\right| \leqslant c\, \|\partial_x^5 \phi\|_{L^2}\, \|\partial_x^2 w\|_{L^2}, \tag{4.31}$$

and

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \partial_x^3 \phi \Lambda_1\, dx\right| &\leqslant c\, \|\partial_x^3 \phi\|_{L^2}\, \|\Lambda_1\|_{L^2}\\ &\leqslant c\, \|\partial_x^3 \phi\|_{L^2}\, [\, \|\partial_x^2 w\|_{L^2}^{1/2}\, \|w\|_{L^2}^{5/2} + \|\partial_x^2 w\|_{L^2}^{5/4}\, \|w\|_{L^2}^{3/4}].\end{aligned} \tag{4.32}$$

On the other hand, using (4.7) we find that

$$\begin{aligned}\Gamma_2^2(t) &\equiv \int_{-\infty}^{\infty} ((\partial_x^2 w)^2 + \tfrac{5}{36}\, w^4 - \tfrac{5}{3} w(\partial_x w)^2)\, dx\\ &\geqslant \|\partial_x^2 w\|_{L^2}^2 - c\, \|\partial_x^2 w\|_{L^2}^{1/2}\, \|w\|_{L^2}^{3/2} - c\, \|\partial_x w\|_{L^4}^2\, \|w\|_{L^2}\\ &\geqslant \tfrac{1}{2}\, \|\partial_x^2 w\|_{L^2}^2 - c_{0T'},\end{aligned} \tag{4.33}$$

which allows us to state that for $t \in [\, T_0/2,\, T'\,]$

$$\|\partial_x^2 w(t)\|_{L^2} \leqslant 2\Gamma_2^2(t) + c_{0T'}. \tag{4.34}$$

Collecting the results in (4.23)–(4.34) we obtain that for $t \in [\, T_0/2,\, T'\,]$

$$\begin{aligned}\frac{d}{dt}\, \Gamma_2^2(t) &\leqslant c\left(\frac{1}{t} + \sum_{j=2}^{3}\, \|\partial_x^j \phi(t)\|_{L^2}\right) \Gamma_2^2(t)\\ &\quad + c_{0T'} \sum_{j=2}^{5}\, \|\partial_x^j \phi(t)\|_{L^2}\, \Gamma_2(t) + c_{0T'}\, \frac{1}{t},\end{aligned} \tag{4.35}$$

which implies that for $t \in [T_0/2, T']$

$$\sup_{[T_0/2,\, T']} \Gamma_2(t) \leqslant c\left( \|w_0\|_{H^2};\ \sum_{j=2}^{5} \|\partial_x^j \phi\|_{L^1_{T'} L^2_x} \right). \tag{4.36}$$

Combining (4.7), (4.21), (4.34), and (4.36) we conclude that for $t \in [T_0/2, T']$

$$\sup_{[T_0/2,\, T']} \|w(t)\|_{H^2} \leqslant c\left( \|w_0\|_{H^2};\ \sum_{j=2}^{5} \|\partial_x^j \phi\|_{L^1_T L^2_x} \right), \tag{4.37}$$

which is the desired *a priori* estimate.

Finally we turn our attention to the uniqueness part. Suppose that $u_1 = u_1(x, t)$ is another solution of the IVP (1.1). Since the solution of the IVP (2.1) $\phi(\cdot)$ is unique we find that $w_1(x, t) = u_1(x, t) - \phi(x, t)$ solves the IVP (3.1). The argument in (3.60)–(3.62) shows that $w_1(x, t) = w(x, t)$, and completes the proof of Theorem 1.2.

## 5. FURTHER RESULTS

In this section we shall state another result concerning the global existence of solutions to the IVP (1.1) with unbounded data. This will follow as a combination of the method of proofs used in [5] for Theorem 1.1, and that given above for Theorem 1.2.

We need the following condition. There exist $k$, $M > 0$ such that

$$\begin{cases} \dfrac{d^j}{dx^j} v(x) = o(|x|^{1-j}), & x \to -\infty, \quad j = 0, 1, ...., 8, \\ \dfrac{d^j}{dx^j} v(x) = 0(|x|^{k-j}), & x \to \infty, \quad\ \ j = 0, 1, ...., 8, \\ \text{with } \dfrac{d}{dx} v(x) > 0, & \text{for} \quad x > M. \end{cases} \tag{5.1}$$

THEOREM 5.1. *If* $u_0 \in C^8(\mathbb{R})$, *and either* $u_0(x)$ *or* $-u_0(-x)$ *satisfy the condition* (5.1), *then the IVP* (1.1) *has a unique global classical solution.*

The idea of the proof of Theorem 5.1 can be abridged as follows. Assume that the initial data $u_0(x)$ satisfies (5.1). Ten for any given $T > 0$ we split the initial data as

$$u_0(x) = w_0(x) + h(x), \tag{5.2}$$

such that $w_0 \in C_0^8(\mathbb{R})$, $h(0)=0$, $h'(x)>0$ for $x>0$, and such that the IVP (2.1) with initial data $h(x)$ has a unique classical solution $\phi(x, t)$ defined in the time interval $[0, T]$. Moreover, using our assumptions (5.1) on $u_0(x)$ we can choose $h(x)$ such that the values of the solution of the solution $\phi(x, t)$ with $x<0$, and $t \in [0, T]$ depends only on the values of $h(x)$, with $x<0$. Roughly, this allows us to solve the IVP (3.1) in the time interval $[0, T]$ by using the estimates for $\phi(x, t)$ deduced in [5] for the case where $x<0$, $t \in [0, T]$, and the argument in the proof of Theorem 3.1 for $x>0$, $t \in [0, T]$.

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